Day 3 - AM

Double Integrals

↳ graphically measure volume of a shape in space

$$\iint (f+g)\,dA = \iint f\,dA + \iint g\,dA$$

$$\iint cf\,dA = c\iint f\,dA$$

Iterated Integrals - integrate one variable, then the other.

Ex: $\int_2^4 \left[\int_1^9 ye^x\,dy\right] dx$

$$= \int_2^4 e^x \left[\left(\tfrac{1}{2}y^2\right)\Big|_1^9\right] dx = \int_2^4 e^x\left(\tfrac{81}{2} - \tfrac{1}{2}\right) dx$$

$$= \int_2^4 40e^x\,dx = 40\int_2^4 e^x\,dx = 40(e^x)\Big|_2^4$$

$$= \boxed{40(e^4 - e^2)}$$

Ex: $\int_1^9 \left[\int_2^4 ye^x\,dx\right] dy$

$$= \int_1^9 y\left(\left[e^x\right]\Big|_2^4\right) dy = \int_1^9 y(e^4 - e^2)\,dy$$

$$= (e^4 - e^2)\int_1^9 y\,dy = (e^4 - e^2)\left(\tfrac{1}{2}y^2\right)\Big|_1^9$$

$$= (e^4 - e^2)\left(\tfrac{81}{2} - \tfrac{1}{2}\right) = \boxed{40(e^4 - e^2)}$$

→

Fubini's Thm

for $f(x,y)$ continuous on $R=[a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Its okay to have functions in the limits of integration!

$$\int_1^3 \int_1^{x^2} xy \quad dy\,dx$$

$$= \int_1^3 x \cdot \left(\frac{1}{2}y^2\right)\Big|_1^{x^2} dx$$

$$= \int_1^3 x\left(\frac{1}{2}x^4 - \frac{1}{2}\right) dx = \int_1^3 \left(\frac{x^5}{2} - \frac{x}{2}\right) dx$$

$$= \left(\frac{1x^6}{12} - \frac{x^2}{4}\right)\Big|_1^3 = \left(\frac{243}{4} - \frac{9}{4}\right) - \left(\frac{1}{12} - \frac{1}{4}\right) = \boxed{\frac{176}{3}}$$

Triple Integrals work the same way!

Sequences

A sequence is any function from the integers to some set, usually $\mathbb{R}$, the set of real numbers.

Ex: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\quad a_n = \frac{1}{n}$

$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ $\quad a_n = 1 - \frac{1}{n}$

$0, -1, 2, -3, 4, \ldots$ $\quad a_n = (-1)^n n$

So $a_n = f(n)$ $\quad$ n is the index

a_n is the n^{th} term

"closed form" - no recursion

Ex: $a_1 = 1 \quad a_n = 2a_{n-1} + 1$ recursive

$a_2 = 2a_1 + 1 = 2(1) + 1 = 3$

$a_3 = 2a_2 + 1 = 2(3) + 1 = 7$

$1, 3, 7, 15, 31, 63, \ldots$

Fibonacci's Sequence is a recursive one!

$F_0 = 1 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2}$

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Does a sequence converge?

Ex: $a_n = \frac{1}{n} \to 0$ as $n \to \infty$

The sequence $\{a_n\}_{n=1}^{\infty}$ has limit L if for any $\varepsilon > 0$ there exists N so that if $n > N$, then $|a_n - L| < \varepsilon$

Claim: $\{a_n\} \to 0$ if $a_n = \frac{1}{n}$

Proof: $|a_n - 0| = |a_n| = \left|\frac{1}{n}\right| < \varepsilon$ provided $n > \frac{1}{\varepsilon}$

$\therefore$ choose $N = \lceil 1/\varepsilon \rceil$ to get the desired effect. ∎

Thm: If $f(n) = a_n$ and $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} a_n = L$

Ex: $\lim_{n \to \infty} \frac{n + \ln n}{n^2}$

"$\frac{\infty}{\infty}$" $\to$ L'Hopital's Rule

$= \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2n} \approx \frac{1 + 0}{\infty} = \boxed{0}$

So $\{a_n\} = \left\{\frac{n + \ln n}{n^2}\right\}_{n=1}^{\infty} \to 0.$

<u>Geometric Sequence</u> $\{a_n\} = \{cr^n\}$ for some c & r

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } 0 \le r < 1 \\ c & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

*See <u>Limit Laws of Sequences Handout</u>!

<u>Claim:</u> $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

<u>Pf:</u> $-1 \le \sin(n) \le 1$

$$\underset{\substack{\downarrow \\ 0}}{-\frac{1}{n}} \le \frac{\sin(n)}{n} \le \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}$$

thus by Squeeze Thm, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$. ∎